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SOLUTION OF FUNDAMENTAL PROBLEMS OF THE THEORY OF ELASTICITY FOR INCOMPRESSIBLE MEDIA*

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Limiting behavior of the solutions of the fundamental problems of the theory of elasticity with the Poisson's ratio $\sigma \to 1/2$ is investigated. It is shown that the limits of the solutions of the fundamental problems are solutions of the corresponding Fredholm equations obtained from the initial equations by passing to the integral operators at $\sigma = 1/2$.

1. Let $\mathbf{u}(\mathbf{x}) = (u_1, u_2, u_3)$ be the displacement vector of the elastic body D filling a part of the space $R^3 \ni \mathbf{x}$ and bounded by a closed Liapunov surface S. The vector $\mathbf{u}(\mathbf{x})$ satisfies, in D, the Lamé equation

$$L_\sigma \mathbf{u} \equiv \Delta \mathbf{u} + (1-2\sigma)^{-1} \operatorname{grad} \operatorname{div} \mathbf{u} = 0$$

We shall assume for simplicity that the surface S is connected, and consider the following four fundamental problems:

Problem $1^\pm$. $L_\sigma \mathbf{u}(\mathbf{x}) = 0, \quad \mathbf{x} \in D^\pm; \quad \mathbf{u}(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in S$.

Problem $2^\pm$. $L_\sigma \mathbf{u}(\mathbf{x}) = 0, \quad \mathbf{x} \in D^\pm; \quad T_{n\sigma}\mathbf{u}(\mathbf{x}) = \mathbf{g}(\mathbf{x}), \quad \mathbf{x} \in S$.

Here D^+ and D^- are the bounded and unbounded part of R^3 with the boundary S, $T_{n\sigma}\mathbf{u}$ is a vector with components

$$(T_{n\sigma}\mathbf{u})_i = \frac{E}{2(1+\sigma)} \sum_{k=1}^{3} \left[\frac{\sigma}{1-2\sigma}\delta_{ik} \operatorname{div}\mathbf{u} + \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right] n_k$$

E is the modulus of elasticity and n_k are the direction cosines of the outward normal $\mathbf{n}$ to S.

Let L_0 be an operator acting on a pair of functions $\mathbf{u}$ and p according to the rule

$$L_0(\mathbf{u};\ p) = \{\eta\Delta\mathbf{u} - \operatorname{grad} p;\ \operatorname{div}\mathbf{u}\}$$

Problem $1_0^\pm$. $L_0(\mathbf{u};\ p) = 0, \quad \mathbf{x} \in D^\pm; \quad \mathbf{u} = \mathbf{f}, \quad \mathbf{x} \in S$.

Problem $2_0^\pm$. $L_0(\mathbf{u};\ p) = 0, \quad \mathbf{x} \in D^\pm; \quad T_{n0}(\mathbf{u};\ p) = \mathbf{g}, \ \mathbf{x} \in S$.

The problems $1_0^\pm$ and $2_0^\pm$ describe a stationary Stokes' flow of a viscous incompressible fluid, while the vector $\mathbf{u}$ has a meaning of velocity, p is the pressure and η is the coefficient of dynamic viscosity. We assume that f and g are twice continuously differentiable on S.

Problem 1^+ always has a solution which is unique, and the problems 1^-, 1_0^-, 2^- and 2_0^- have unique solutions in the class of functions with an asymptotics at infinity $1/|x|$. The problem 1_0^+ has not more than one solution, and the solution exists only when $(\mathbf{f}, \mathbf{n}) = 0$. Here $(\cdot,\cdot)$ denotes a scalar product in $L_2(S)$. The Problems 2^+ and 2_0^+ can be solved if and only if $(\mathbf{g}, \psi_i) = 0$ $(i = 1, 2, \ldots, 6)$, and the solutions are defined with an accuracy of up to a linear combination of the vectors ψ_i (here ψ_i denote the linearly independent vectors of inelastic displacement). The solutions of the problems $1^\pm, 2^\pm, 1_0^\pm$ and $2_0^\pm$ are all twice continously differentiable in $D^\pm$ (see /1,2/).

2. Let

$$\mathbf{V} = \{V_{ik}\}_{i,k=1}^3 = \frac{3}{8\pi E(1-\sigma)}\left\{\frac{3-4\sigma}{|\mathbf{x}-\mathbf{y}|}\delta_{ik} + \frac{(y_i - x_i)(y_k - x_k)}{|\mathbf{x}-\mathbf{y}|^3}\right\}$$

be the fundamental solution of the operator $L_\sigma : L_{\sigma x}\mathbf{V}(\mathbf{x}, \mathbf{y}) = -2\delta(\mathbf{x}-\mathbf{y}) I$ (I is the unit matrix) and the pair

$$\mathbf{V}_0 = \{V_{0ik}\} = \frac{3}{4\pi E}\left\{\frac{\delta_{ik}}{|\mathbf{x}-\mathbf{y}|} + \frac{(x_i - y_i)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^3}\right\}$$

$$\{P^k(\mathbf{x}, \mathbf{y})\}_{k=1}^3 = \frac{1}{2\pi}\left\{\frac{x_k - y_k}{|\mathbf{x}-\mathbf{y}|^3}\right\}_{k=1}^3$$

be the fundamental solution of the operator $L_0 : L_0(\mathbf{V}_0;\ P) = \{-2\delta(\mathbf{x}-\mathbf{y}); 0\}$ for $\eta = E/3$. Further, setting $\sigma = 1/2 - \varepsilon$, we shall denote the symbols referring to the Problems $1^\pm$ and $2^\pm$ at the particular value of σ, by the subscript ε. We shall also utilize the following

*Prikl.Matem.Mekhan.,44,No.5,867-874,1980 expressions for the potentials of density $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ (the prime denotes a transposition)

$$\Pi(\mathbf{x}, \varphi) = \int_S \sum_{k=1}^{3} P^k(\mathbf{x}, \mathbf{y}) \varphi^k(\mathbf{y}) \, d_yS, \quad \Pi_1(\mathbf{x}, \varphi) = \frac{E}{6\pi} \frac{\partial}{\partial x_j} \int_S \frac{x_k - y_k}{|\mathbf{x} - \mathbf{y}|^3} \varphi_k(\mathbf{y}) n_j(\mathbf{y}) \, d_yS$$

$$\mathbf{W}_\varepsilon(\mathbf{x}, \varphi) = \int_S [T_{n\varepsilon y} V_\varepsilon(\mathbf{x}, \mathbf{y})]' \varphi(\mathbf{y}) \, d_yS, \quad \mathbf{W}_0(\mathbf{x}, \varphi) = \int_S [T_{n0y}(\mathbf{V}_0(\mathbf{x}, \mathbf{y}); P^k(\mathbf{x}, y))]' \varphi(\mathbf{y}) \, d_yS$$

Let us define the operators acting in $L_2(S)$ by the equations

$$T_\varepsilon \varphi = \int_S T_{n\varepsilon x} \mathbf{V}_\varepsilon(\mathbf{x}, \mathbf{y}) \varphi(\mathbf{y}) \, d_yS, \quad T_\varepsilon^* \varphi = \int_S [T_{n\varepsilon y} \mathbf{V}_\varepsilon(\mathbf{x}, \mathbf{y})]' \varphi(\mathbf{y}) \, d_yS$$

$$T_0 \varphi = \int_S T_{n0x}(\mathbf{V}(\mathbf{x}, \mathbf{y}); P^k(\mathbf{x}, \mathbf{y})) \varphi(\mathbf{y}) \, d_yS, \quad T_0^* \varphi = \int_S [T_{n0y}(\mathbf{V}_0(\mathbf{x}, \mathbf{y}); P^k(\mathbf{x}, \mathbf{y}))]' \varphi(\mathbf{y}) \, d_yS, \quad \mathbf{x} \in S$$

The properties of the operators T_ε and T_ε^* have been investigated in /2/, and those of T_0 and T_0^*, in /3/. The operators T_ε and T_ε^* are conjugated in $L_2(S)$ and continuous, the operators T_0 and T_0^* are conjugated in $L_2(S)$ and completely continuous, $\Sigma(T_\varepsilon) \ni -1$, $\Sigma(T_0) \ni 1, -1$ and $\delta > 0$, $\delta_1 > 0$ exists such that

$$\Sigma(T_\varepsilon) \setminus \{-1\} \subset [-1 + \delta, 1 - \delta], \quad \Sigma(T_0) \setminus \{-1, 1\} \subset [-1 + \delta_1, \ 1 - \delta_1], \ N(I + T_0^*) = N(I + T_\varepsilon^*) = \{\psi_i\}_{i=1}^6, \ \forall \varepsilon > 0$$

Here $\Sigma(\cdot)$ and $N(\cdot)$ denote, respectively, the spectral manifold and the kernel of the operator within the brackets. It can be shown directly that $T_\varepsilon = T_0 + 2\varepsilon(1 + 2\varepsilon)^{-1} T_1$ where T_1 is independent of ε.

3. It is natural to require that the state of stress of an incompressible body a) shows little change when σ deviates from ½ by a small amount, and b) depends continuously on the boundary conditions. Clearly, we cannot have more than one solution satisfying the condition a) (in the case of the Problem 2^+ the solution under consideration is accurate to within the inelastic displacement). This follows from the uniqueness of the solution of the corresponding boundary value problem for $\varepsilon > 0$. When we say a solution of the boundary value problem of the theory of elasticity for an incompressible body, we mean the limit of the solution of the corresponding boundary value problem as $\varepsilon \to 0$ $(\varepsilon > 0)$. Obviously, a solution defined in this manner satisfies the condition a). Below we shall show that the above limit exists for the Problems $1^\pm$ and $2^\pm$, and next we shall prove that this implies the fulfilment of condition b).

Let us write the boundary value problem in the form:

$$A_\varepsilon \mathbf{u}_\varepsilon = \mathbf{F}; \quad A_\varepsilon \equiv \{L_\sigma; \ \gamma\}, \quad \mathbf{F} \equiv \{0, \ \mathbf{f}\}$$

Here γ is the corresponding boundary operator of the boundary value problem. For the fundamental problems $1^\pm$ and $2^\pm$ the operator A_ε^{-1} is defined in $R(A_\varepsilon)$ and continuous ($R(\cdot)$ denotes the domain of values of the operator within the brackets). If the limit

$$\mathbf{u}_0 = \lim \mathbf{u}_\varepsilon = \lim A_\varepsilon^{-1} \mathbf{F} \ (\varepsilon \to 0) \tag{3.1}$$

exists for every $\mathbf{F} \in R(A_\varepsilon)$, then from the Banach—Steinhaus principle of uniform boundedness it follows that the sequence A_ε^{-1} is bounded uniformly in ε and the operator A_0^{-1} defined by means of (3.1) is bounded, and this implies that $\mathbf{u}_0$ depends continuously on $\mathbf{F}$.

4. Next we shall turn our attention to finding the limits of the solutions of the fundamental problems with $\sigma \to 1/2$.

Problem 2^+. The solution is sought in the form of the potential of a simple layer, i.e. we carry out the substitution

$$\mathbf{u}_\varepsilon = \int_S \mathbf{V}_\varepsilon(\mathbf{x}, \mathbf{y}) \varphi_\varepsilon(\mathbf{y}) \, d_yS \tag{4.1}$$

The substitution is equivalent and reduces the problem to an equation in φ_ε on the boundary S. This equation can be written in the operator form as

$$\varphi_\varepsilon + T_\varepsilon \varphi_\varepsilon = \mathbf{g} \tag{4.2}$$

A solution of (4.2) exists if and only if $\mathbf{g} \in R(I + T_\varepsilon) = {}^\perp N(I + T_\varepsilon^*)$. The operator $(I + T_\varepsilon)^{-1}$ regarded as the value of the resolvent of the constriction of the operator T_ε on $R(I + T_\varepsilon)$ at the point -1, is defined and continuous in $R(I + T_\varepsilon)$. In addition, from the properties of $\Sigma(T_\varepsilon)$ if follows that

$$(I + T_\varepsilon)^{-1} = \sum_{k=0}^{\infty} (-T_\varepsilon)^k$$

and any solution of (4.2) can therefore be written in the form

$$\varphi_\varepsilon = \sum_{k=0}^{\infty} (-T_\varepsilon)^k g + \Phi \tag{4.3}$$

Here we have $\Phi \in N(I + T_\varepsilon)$ and the potential of a simple layer of density Φ is a vector of inelastic displacement of D^+. Let us substitute (4.3) into (4.1) and make ε tend to zero ($\sigma \to 1/2$). Since T_ε as an operator function of ε and $\mathbf{V}_\varepsilon(\mathbf{x}, \mathbf{y})$ as a function of ε are both continuous at the zero, we have

$$\| \mathbf{u}_\varepsilon - \mathbf{u}_0 \|_{L_2} \to 0 \ (\varepsilon \to 0)$$

where $\mathbf{u}_0$ is the potential of a simple layer of density

$$\mathbf{u}_0 = \int_S \mathbf{V}_0(\mathbf{x}, \mathbf{y}) \varphi_0(\mathbf{y}) \, d_y S$$

The pair $(\mathbf{u}_0; p)$ where $p = \Pi(\mathbf{x}, \varphi)$, satisfies the boundary value Problem 2_0^+.

Problem 1^-. We seek the solution in the form

$$\mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{W}_\varepsilon(\mathbf{x}, \varphi) + \sum_{i=1}^{6} c_i \int_S \mathbf{V}_\varepsilon(\mathbf{x}, \mathbf{y}) \psi_i(\mathbf{y}) \, d_y S \tag{4.4}$$

The function (4.4) satisfies in D^- the condition $L\mathbf{u} = 0$, and the condition on S leads to the equation

$$\varphi + T_\varepsilon^* \varphi = \mathbf{F} \equiv \mathbf{f}(\mathbf{x}) - \sum_{i=1}^{6} c_i \int_S \mathbf{V}_\varepsilon(\mathbf{x}, \mathbf{y}) \psi_i(\mathbf{y}) \, d_y S \tag{4.5}$$

Let $\varphi_i(x)$ $(i = 1, 2, \ldots, 6)$ be the eigenfunctions corresponding to the value of the operator T_ε equal to -1. If c_i can be chosen so that the conditions of solvability of (4.5) $(\mathbf{F}, \varphi_i) = 0$ $(i = 1, 2, \ldots, 6)$, hold, then the solution of the initial problem can be found uniquely since the potential of the double layer of density ψ_i is zero when $\mathbf{x} \in D^-$.

Let us consider, instead of (4.5), the equation

$$\varphi_\varepsilon + Q_\varepsilon \varphi_\varepsilon = \mathbf{F}, \quad Q_\varepsilon = T_\varepsilon^* - \sum_{i=1}^{6} \psi_i(\cdot, \psi_i) \tag{4.6}$$

It was shown in /4/ that under the conditions satisfying the operator T_ε we have a) $\Sigma(Q_\varepsilon) \subset \Sigma(T_\varepsilon) \setminus \{-1\}$, therefore the equation (4.6) has a solution for any $\mathbf{F}$, b) if φ is a solution of (4.6), then the conditions $(\varphi, \psi_i) = 0$ and $(\mathbf{F}, \varphi_i) = 0$ $(i = 1, 2, \ldots, 6)$ follow from each other. Therefore if any of these conditions hold, then the solution φ is also a solution of (4.5).

Let R_ε denote the value of the resolvent of Q_ε at the point -1. By virtue of the properties of the operator T_ε^* we have

$$R_\varepsilon = \sum_{k=0}^{\infty} (-Q_\varepsilon)^k$$

The requirement that $(\varphi, \psi_i) = 0$ $(i = 1, 2, \ldots, 6)$, yields a system of linear algebraic equations and their solution gives c_i. It remains to show that the determinant of the matrix of the coefficients accompanying c_i is not zero. We have

$$(\mathbf{F}, \varphi_i) = (\mathbf{f}, \varphi_i) - \sum_{j=1}^{6} c_j (\psi_j, \psi_i)$$

which follow from the symmetry of $\mathbf{V}_\varepsilon(\mathbf{x}, \mathbf{y})$ and the relations

$$\int_S \mathbf{V}_\varepsilon(\mathbf{x}, \mathbf{y}) \varphi_i(\mathbf{y}) \, d_y S = \psi_i(\mathbf{x}) \quad (i = 1, 2, \ldots, 6)$$

Consequently c_i are determined uniquely by the conditions $(\mathbf{F}, \varphi_i) = 0$ and hence from the conditions $(\varphi, \psi_i) = 0$. Thus the equation (4.5) has a corresponding function

$$\varphi_\varepsilon = R_\varepsilon \mathbf{f} - \sum_{i=1}^{6} c_i R_\varepsilon \int_S \mathbf{V}_\varepsilon(\mathbf{x}, \mathbf{y}) \psi_i(\mathbf{y}) \, d_y S$$

The coefficients c_i represent a solution of the system $(\varphi, \psi_i) = 0$ $(i = 1, 2, \ldots, 6)$. Repeating the previous arguments for $\varepsilon = 0$, we obtain a component of the solution $\mathbf{u}_0$ of the Problem 1_0^- in the form (4.5). By virtue of the continuity in ε at zero and of the continuity of $\mathbf{V}_\varepsilon$ in ε, we conclude that $\| \mathbf{u}_\varepsilon - \mathbf{u}_0 \|_{L_2(D^-)} \to 0$ as $\varepsilon \to 0$. Here the pair $(\mathbf{u}_0; p)$ where $p = \Pi_1(x, \varphi)$ satisfies the boundary value Problem 1_0^-.

Problem 1^+. The following assertion was proved in /5/. The solution of the problem

$$L_\varepsilon \mathbf{u}_\varepsilon = \frac{3-\varepsilon}{E}\mathbf{F}, \quad \mathbf{x}\in D^+, \quad \mathbf{u}_\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x}\in S \quad (\mathbf{F}\in L_2(D^+))$$

can be written in the form of the following series converging on the norm $W_2^1(D^+)$:

$$\mathbf{u}_\varepsilon = \sum_{k=0}^{\infty}(2\varepsilon)^k \mathbf{u}_k$$

where $\mathbf{u}_0$ is a solution of the problem

$$L_0(\mathbf{u}_0; p) = \frac{3}{E}\mathbf{F}, \quad \mathbf{x}\in D^+; \quad \mathbf{u}_0(\mathbf{x}) = 0, \quad \mathbf{x}\in S$$

This implies, in particular, the following relation which will be used later:

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(S)} \leqslant c\,\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{W_2^1(D^+)} = O(\varepsilon) \tag{4.7}$$

It can be shown that the relation (4.7) also holds for the solutions of the Problems 1^+ and 1_0^+. To do this we first note the fact that the deformation energy of an incompressible body is finite requires that the condition $\operatorname{div}\mathbf{u} = 0$ and hence $(\mathbf{u}|_S, n) = 0$ holds. We shall therefore assume that the condition in the problem of the state of stress in an incompressible body, holds.

Let us consider the problems

$$L_\varepsilon \mathbf{u}_\varepsilon = 0, \ \mathbf{x}\in D^+; \ \mathbf{u}_\varepsilon = \mathbf{f}, \ \mathbf{x}\in S \tag{4.8}$$

$$L_0(\mathbf{u}_0; p) = 0, \ \mathbf{x}\in D^+; \ \mathbf{u}_0 = \mathbf{f}, \ \mathbf{x}\in S \tag{4.9}$$

We extend $\mathbf{f}\in C^3(S)$ to $\Phi\in W_2^2(D^+)$, such that $\operatorname{div}\Phi = 0$. Carrying out the substitutions $\mathbf{u}_\varepsilon = \mathbf{u}_{1\varepsilon} + \Phi$, $\mathbf{u}_0 = \mathbf{u}_{10} + \Phi$, we arrive at the problems

$$L_\varepsilon \mathbf{u}_{1\varepsilon} = -\Delta\Phi, \ \mathbf{x}\in D^+; \ \mathbf{u}_\varepsilon = 0, \ \mathbf{x}\in S$$
$$L_0(\mathbf{u}_{10}; p) = -\Delta\Phi, \ \mathbf{x}\in D^+; \ \mathbf{u}_0 = 0, \ \mathbf{x}\in S$$

It is clear now that (4.7) holds also for the solutions of the problems (4.8) and (4.9).

Problem 2^-. The problem can be solved for $\varepsilon > 0$, starting from an integral equation for the displacement at the boundary obtained from the Green—Betti formula

$$-\mathbf{u}_\varepsilon + T_\varepsilon^*\mathbf{u}_\varepsilon = -\int_S V_\varepsilon(\mathbf{x},\mathbf{y})\, g(\mathbf{y})\, d_yS \equiv G_\varepsilon(\mathbf{x}) \tag{4.10}$$

To show that the limit $\lim u_\varepsilon$ as $\varepsilon\to 0$ exists, we must investigate the structure of the resolvent $(I + T_\varepsilon^*)^{-1}$ as $\varepsilon\to 0$. First we shall make a number of comments concerning the operator T_0^* and the Problems 1_0^+ and 2_0^-, supplementing the facts which are already known.

Let us seek a solution of the Problem 1_0^+ in the form of the potential of a double layer

$$\mathbf{u}_0 = \mathbf{W}_0(\mathbf{x}, \varphi), \quad p = \Pi(\mathbf{x}, \varphi)$$

We have the following integral equation for the vector φ at the boundary:

$$-\varphi + T_0^*\varphi = \mathbf{f} \tag{4.11}$$

Equation (4.11) has a solution when $\mathbf{f}\in R(-I + T_0^*) = {}^\perp N(-I + T_0) = \{\mathbf{f} : (\mathbf{f}, \mathbf{n}) = 0\}$, and the solution can be written in the form

$$\varphi = \varphi_0 + c\mu, \quad \varphi_0 = -\frac{1}{2}\mathbf{f} - \frac{1}{2}\sum_{k=0}^{\infty} T_0^{*k}(I + T_0^*)\mathbf{f} \in {}^\perp N(-I + T_0) \tag{4.12}$$

where c is an arbitrary constant and μ is a solution of the equation $-\mu + T_0^*\mu = 0$. From the uniqueness of the solution of the problem we conclude that

$$\mathbf{W}_0(\mathbf{x}, \mu) = 0, \quad \Pi_1(\mathbf{x}, \mu) = \text{const}, \quad \mathbf{x}\in D^+$$

Since the relation $\operatorname{div}_x \mathbf{V}_0(\mathbf{x},\mathbf{y}) = 0$, $\mathbf{x}\in D^\pm$, $\mathbf{y}\in S$ holds for the potential $\mathbf{V}_0(\mathbf{x},\mathbf{y})$, we have, for any value of $\mathbf{g}$,

$$\Big(\int_S \mathbf{V}_0(\mathbf{x},\mathbf{y})\,\mathbf{g}(\mathbf{y})\, d_yS\,|_{\mathbf{x}\in S}, \mathbf{n}\Big) = 0 \tag{4.13}$$

Let us now return to the Problem 2^-. Using the Green—Betti formula for the Problem 2_0^- and the fundamental solution $\mathbf{V}_0 P^k$, we obtain, in the course of the passage to S, the following integral equation for $\mathbf{u}$ at the boundary of S:

$$-\mathbf{u} + T_0^*\mathbf{u} = -\int_S \mathbf{V}_0(\mathbf{x},\mathbf{y})\,\mathbf{g}(\mathbf{y})\, d_yS \equiv \mathbf{G}_0(\mathbf{x}) \tag{4.14}$$

Equation (4.14) has a solution by virtue of (4.13), at any value of g, and the solution can be written in the form analogous to (4.12)

$$\mathbf{u} = \mathbf{u}_0 + c\boldsymbol{\mu} \tag{4.15}$$

The solution of the Problem 2_0^- must satisfy the condition $\operatorname{div} \mathbf{u} = 0$, $x \in D^-$ or $(\mathbf{u}|_S, \mathbf{n}) = 0$. Since $(\mathbf{u}_0, \mathbf{n}) = 0$ and by virtue of the fact that unity is a simple eigenvalue of the operator T_0, $(\boldsymbol{\mu}, \mathbf{n}) \neq 0$, we have $c = 0$. Thus the solution of the Problem is the pair $(\mathbf{u}_0; p)$ where

$$\mathbf{u}_0 = -\frac{1}{2}\mathbf{G}_0 - \frac{1}{2}\sum_{k=0}^{\infty} T_0^{*k}(I + T_0^*)\mathbf{G}_0$$

Let us inspect the resolvent of the operator T_ε^* $R(\lambda, T_\varepsilon^*) = (\lambda I - T_\varepsilon^*)^{-1}$. The operator T_ε^* has an eigenvalue of total multiplicity equal to unity, of the form $1 + \eta(\varepsilon)$ where $\eta(\varepsilon)$ is an analytic function in the neighborhood of $\varepsilon = 0$ and $\eta(0) = 0$. The following representation holds near the point $1 + \eta(\varepsilon)$ (see e.g. /6/):

$$R(\lambda, T_\varepsilon^*) = \frac{P(\varepsilon)}{\lambda - 1 - \eta(\varepsilon)} + R_0(\lambda, T_\varepsilon^*) \tag{4.16}$$

Here $P(\varepsilon)$ is a projector which can be written in the form $P(\varepsilon) = \boldsymbol{\mu}(\varepsilon)(\cdot, \mathbf{n}(\varepsilon))$ where $\boldsymbol{\mu}(\varepsilon) = \boldsymbol{\mu} + \varepsilon\boldsymbol{\mu}_1 + \ldots$, $\mathbf{n}(\varepsilon) = \mathbf{n} + \varepsilon\mathbf{n}_1 + \ldots$ are the eigenvalues of the operators T_ε^* and T_ε respectively, corresponding to the eigenvalue $1 + \eta(\varepsilon)$ and analytic in the neighborhood of $\varepsilon = 0$; $R_0(\lambda, T_\varepsilon^*)$ is the operator-valued function analytic near the point $\lambda = 1 + \eta(\varepsilon)$.

It can be shown that $\lim R_0(1, T_\varepsilon^*)\mathbf{G}_\varepsilon = R_0(1, T_0^*)\mathbf{G}_0$ $(\varepsilon \to 0)$ in the strict sense. Indeed, let $\mathbf{G}_\varepsilon = \mathbf{G}_{1\varepsilon} + \mathbf{G}_{2\varepsilon}$ where $\mathbf{G}_{2\varepsilon} = P(\varepsilon)\mathbf{G}_\varepsilon$ and $P(\varepsilon)\mathbf{G}_{1\varepsilon} = 0$. From the definition of the operator $R_0(1, T_\varepsilon^*)$ we have

$$R_0(1, T_\varepsilon^*)\mathbf{G}_\varepsilon = R(1, T_\varepsilon^*)\mathbf{G}_{1\varepsilon} = -\frac{1}{2}\mathbf{G}_{1\varepsilon} - \frac{1}{2}\sum_{k=0}^{\infty} T_\varepsilon^{*k}(I + T_\varepsilon^*)\mathbf{G}_{1\varepsilon}, \quad R_0(1, T_0^*)\mathbf{G}_0 = -\frac{1}{2}\mathbf{G}_0 - \frac{1}{2}\sum_{k=0}^{\infty} T_0^{*k}(I + T_0^*)\mathbf{G}_0$$

and from this follows

$$\|R_0(1, T_0^*)\mathbf{G}_0 - R_0(1, T_\varepsilon^*)\mathbf{G}_\varepsilon\| \leqslant \frac{1}{2}\|\mathbf{G}_{1\varepsilon} - \mathbf{G}_0\| + \frac{1}{2}\left\|\sum_{k=0}^{N} T_0^*(I + T_0^*)\mathbf{G}_0 - \sum_{k=0}^{N} T_\varepsilon^{*k}(I + T_\varepsilon^*)\mathbf{G}_{1\varepsilon}\right\| + \frac{1}{2}\left\|\sum_{k=N+1}^{\infty} T_0^{*k}(I + T_0^*)\mathbf{G}_0\right\| +$$

$$\frac{1}{2}\left\|\sum_{k=N+1}^{\infty} T_\varepsilon^{*k}(I + T_\varepsilon)\mathbf{G}_{1\varepsilon}\right\|$$

We shall show that the last term tends to zero as $N \to \infty$ uniformly in ε. This is obviously sufficient to ensure that the expression tends to zero from the left as $\varepsilon \to 0$. Let us inspect the contraction of the operator T_ε^* on $M = \{\boldsymbol{\varphi} \in L_2(S): (\boldsymbol{\varphi}, \boldsymbol{\psi}_i) = 0,\ i = 1, 2, \ldots, 6\}$. The spectral radius of this contraction is $\rho(\varepsilon) < 1 - \delta(\varepsilon)$. Moreover, δ_0 exists such that $\delta(\varepsilon) > \delta_0 > 0$, $\forall \varepsilon \in [0, \varepsilon_1]$ where ε_1 is sufficiently small. It is sufficient now to note that $(I + T_\varepsilon^*)\mathbf{G}_{1\varepsilon} \in M$ in order to obtain the estimate

$$\left\|\sum_{k=N+1}^{\infty} T_\varepsilon^{*k}(I + T_\varepsilon^*)\mathbf{G}_{1\varepsilon}\right\| \leqslant (1 - \delta_0)^N \text{ const}$$

Next we turn our attention to the first term of the expansion (4.16). The expression for $P(\varepsilon)$ and by virtue of $(\mathbf{G}_0, \mathbf{n}) = 0$ (see (4.13)) we have $P(\varepsilon)\mathbf{G}_\varepsilon = \varepsilon\boldsymbol{\mu}(\mathbf{G}_0, \mathbf{n}_1) + O(\varepsilon^2)$. Consequently, the sufficient condition for the relation (4.15) to hold for the solution of the Problem 2^- is, that the condition $\lim \eta(\varepsilon)/\varepsilon = \text{const} \neq 0$, $\varepsilon \to 0$ holds. We find that $\eta(\varepsilon) = \eta_k\varepsilon^k + O(\varepsilon^{k+1})$, $k \leqslant 1$. Indeed, let us consider the possibilities: a) $\boldsymbol{\mu}_1 = 0$; b) $\boldsymbol{\mu}_1 \neq 0$. In the case a) we obtain $(T_0^* + \varkappa T_1)\boldsymbol{\mu} = \boldsymbol{\mu}$ $(\varkappa = 2\varepsilon(1 + 2\varepsilon)^{-1})$, i.e. $1 \in \Sigma(T_\varepsilon^*)$ and this is not possible, therefore $\boldsymbol{\mu}_1 \neq 0$.

Let $\boldsymbol{\varphi}_\varepsilon$ be a solution of the equation $-\boldsymbol{\varphi}_\varepsilon + T_\varepsilon^*\boldsymbol{\varphi}_\varepsilon = \mathbf{f}$ when $(\mathbf{f}, \mathbf{n}) = 0$

$$\boldsymbol{\varphi}_\varepsilon = \boldsymbol{\varphi}_0 + P(\varepsilon)\mathbf{f}/\eta(\varepsilon) + O(\varepsilon) = \boldsymbol{\varphi}_0 + \varepsilon^{-k+1}\boldsymbol{\mu}(\mathbf{n}_1, \mathbf{f}) + \varepsilon^{-k+2}[(\mathbf{n}_2, \mathbf{f})\boldsymbol{\mu} + (\mathbf{n}_1, \mathbf{f})\boldsymbol{\mu}_1] + O(\varepsilon^{-k+3}) + O(\varepsilon)$$

Consider the potential $\mathbf{W}_\varepsilon(\mathbf{x}, \boldsymbol{\varphi}_\varepsilon)$ for $\mathbf{x} \in D^+$. The previous arguments and the equality $\mathbf{W}_0(\mathbf{x}, \boldsymbol{\mu}) = 0$ together yield

$$\mathbf{W}_\varepsilon(\mathbf{x}, \boldsymbol{\varphi}_\varepsilon) = \mathbf{W}_0(\mathbf{x}, \boldsymbol{\varphi}_0) + \varepsilon^{-k+2}(\mathbf{n}_1, \mathbf{f})\mathbf{W}_0(\mathbf{x}, \boldsymbol{\mu}_1) + O(\varepsilon) + O(\varepsilon^{-k+3}) \tag{4.17}$$

If $\boldsymbol{\mu}_1 = \boldsymbol{\mu}$ (or $\mathbf{n}_1 = \mathbf{n}$), then $(T_0^* + \varkappa T_1)\boldsymbol{\mu} = (1 + \varkappa\eta_1)\boldsymbol{\mu}$ and since $1 \,\overline{\in}\, \Sigma(T_\varepsilon^*)$, we have $\eta_1 \neq 0$ (or $k = 1$ which is the same). If on the other hand $\boldsymbol{\mu}_1 \neq \boldsymbol{\mu}$, then by virtue of the fact that $\boldsymbol{\mu}_1 \,\overline{\in}\, N(-I + T_0^*) = \{\boldsymbol{\mu}\}$, we have $\mathbf{W}_0(\mathbf{x}, \boldsymbol{\mu}_1) \neq \mathbf{W}_0(\mathbf{x}, \boldsymbol{\mu}) = 0$. Moreover, if $\mathbf{n}_1 \neq \mathbf{n}$, then for the "general position" function $\mathbf{f}$ satisfying the condition $(\mathbf{f}, \mathbf{n}) = 0$ only we have $(\mathbf{f}, \mathbf{n}_1) \neq 0$. Since $\mathbf{W}_0(\mathbf{x}, \boldsymbol{\varphi}_0)$ is a solution of the Problem 1_0^+ for $\mathbf{u}_0|_S = \mathbf{f}$, we find from (4.7) and (4.17) that $k = 1$. Thus on the surface S we have

$$\mathbf{u}_\varepsilon = P(\varepsilon)\mathbf{G}_\varepsilon / \eta(\varepsilon) + \mathbf{u}_0 + O(\varepsilon) = A\mu + \mathbf{u}_0 + O(\varepsilon), \quad A = \text{const}$$

In fact, $A = 0$ and $\|\mathbf{u}_\varepsilon - \mathbf{u}_0\| = O(\varepsilon)$.

Indeed, let $\mathbf{u}^\circ(\mathbf{x}) = \lim \mathbf{u}_\varepsilon(\mathbf{x}),\ \mathbf{x} \in D^-,\ \varepsilon \to 0$. Passing in the representation $\mathbf{u}_\varepsilon(\mathbf{x}),\ \mathbf{x} \in D^-$, to the limit through the sum of potentials of the double layer of density $\mathbf{u}_\varepsilon(\mathbf{x}),\ \mathbf{x} \in S$ and a simple layer of density $\mathbf{f}$, we find that $\operatorname{div} \mathbf{u}^\circ(\mathbf{x}) = 0,\ \mathbf{x} \in D^-$. From this we have $(\mathbf{u}^\circ|_S, \mathbf{n}) = 0$ and, as $(\mathbf{u}_0, \mathbf{n}) = 0$, we have $A = 0$.

Thus the limits of the solutions of the four fundamental problems in question exist. From the point of view of the numerical computations, it is important that these limits can be found using the method of successive approximations.

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